SIBC Formulation for a Low-Dispersion Finite Volume Method in the Time Domain

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Surface Impedance Boundary Conditions (SIBC) for the computation of resistive wall wakefields in linear accelerators are developed. The method extends the Staggered Finite Volume Method in the Time Domain (SFVTD) for the discretization of Maxwells equations. It uses an Auxiliary Differential Equation (ADE) formulation for general impedance functions describing frequency dependent wall conductivity, surface roughness or metal oxidation. For the time discretization of the resulting dispersive equations a particular technique based on exponential integration is employed. This allows to preserve the basic properties of the SFVTD method such as low numerical dispersion and optimal stability bound which are of crucial importance for electromagnetic wakefield computations.

***Index Terms*—Computational electromagnetics, Finite volume methods, Surface impedance, Linear particle accelerator.**

I. Introduction

Electromagnetic wakefields due to the finite conductivity of cavity walls are one of the main concerns in the design of electron accelerators. These so called resistive wall wakefields are the largest contributor to beam coupling impedances in the high energy sections of the accelerator where extremely short electron bunches are operated [1]. For an estimation of these contributions one relies (almost) exclusively on numerical simulations, since wakefield measurements within the high-vacuum accelerator chamber are very cumbersome.

Conventional methods for the solution of Maxwell's equations in the time domain, however, will usually fail for this class of problems. This is primary due to the extremely high frequency of wakefields resulting in large numerical dispersion errors. These errors tend to accumulate in the course of the simulation as, e.g., short electron bunches of μm-length propagate over several meters within the accelerator. To cope with this problem, specialized low-dispersion techniques have been proposed [2]. The SFVTD method introduced in [3] is one of them. It represents a volume-integral based formulation with very appealing numerical properties. The dispersion error of SFVTD is substantially smaller than that of the conventional FDTD technique. The crucial property, however, is that the method can be operated at a maximum stable time step corresponding to the 1D-CFL stability limit. Applying SFVTD at this so called 'magic' time step provides the exact, dispersion-free solution for all electromagnetic waves propagating along the three main axis directions (cf. [3]).

In order to take into account resistive and/or rough wall wakefields in such simulations, however, an appropriate implementation of broadband SIBC for SFVTD is needed. In the following, this task is accomplished by combining the ADE technique with a particular time stepping scheme. This latter allows to maintain the numerical dispersion and stability properties of the original SFVTD which are necessary for this type of simulations.

II. The SFVTD Method

The basic idea of the SFVTD discretization is depicted in Fig. 1. Fields and currents are allocated component-wise on the faces of a Cartesian mesh. For each of these components a unique control volume enclosing the corresponding mesh face is introduced. Alternatively, one may think of three secondary, staggered meshes which are obtained by shifting the original mesh along the x-, y- and z-directions, respectively.

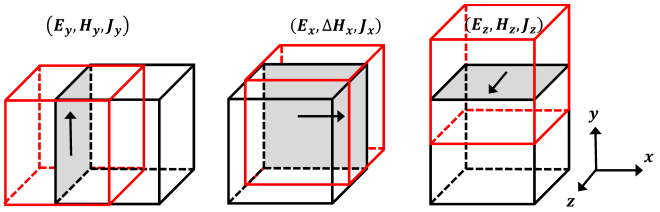

Figure 1: Allocation of fields and currents on mesh cells (black) and corresponding control volumes (red) of SFVTD.

A discretization of Maxwell's equations is obtained by applying the generalized Stokes' theorem for each of the 6 field components on the corresponding control volumes. A detailed derivation of these equations is given in [3]. Here, we begin with the semi-discrete form of SFVTD:

$$\boldsymbol{M}_\mu \frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} = -\boldsymbol{C}\boldsymbol{e} \tag{1}$$

$$\boldsymbol{M}_\epsilon \frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t} = \boldsymbol{C}^T\boldsymbol{h} - \boldsymbol{j} \tag{2}$$

In (1), (2), $\boldsymbol{e}$, $\boldsymbol{h}$ and $\boldsymbol{j}$ represent volume averages over the control volumes of the electric, magnetic and current field components, respectively. The matrices, $\boldsymbol{C}$, $\boldsymbol{M}_\epsilon$ and $\boldsymbol{M}_\mu$ are the *curl*- and *mass*-operators of the method resulting from this particular choice of integration volumes on the mesh. The particular form of these matrices determines the numerical properties of the SFVTD method such as the low numerical dispersion and the large bound of stability.

III. SIBC-ADE Formulation

In the following, first order Leontovich type SIBC are considered, $\boldsymbol{n}\times\boldsymbol{E}(\omega) = Z_s(\omega)\,\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{H}(\omega)$, where $\boldsymbol{n}$ is the normal to the surface. The frequency dependent impedance function $Z_s(\omega)$ is represented by a general pole-residue expansion as

$$Z_s(\omega) = j\omega L + \alpha_0 + \sum_{i=1}^{N_p} \frac{\alpha_i}{j\omega + \beta_i}. \tag{3}$$

In (3), α_i, β_i and L are real coefficients and N_p is the order of the pole-residue model. Note that, the parameter L corresponds to the effective wall inductance which becomes particularly important for rough or oxidized surfaces. Following the procedure proposed in [4], the SIBC is written in the time domain as a set of ADEs,

$$\boldsymbol{n}\times\boldsymbol{E} = \left(\alpha_0 + L\frac{\partial}{\partial t}\right)(\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{H}) + \sum_{i=1}^{N_p}\boldsymbol{G}_i \tag{4}$$

$$\frac{\partial \boldsymbol{G}_i}{\partial t} + \beta_i \boldsymbol{G}_i = \alpha_i(\boldsymbol{n}\times\boldsymbol{n}\times\boldsymbol{H}), \quad i = 1,\ldots,N_p, \tag{5}$$

where $\boldsymbol{G}_i$ are auxilliary fields resembling effective magnetic currents on SIBC surfaces. Imposing (4) on SIBC surfaces and applying SFVTD discretization leads to a modified semi-discrete Faraday's law:

$$(\boldsymbol{M}_\mu + L\boldsymbol{A}_s)\frac{\mathrm{d}\boldsymbol{h}}{\mathrm{d}t} + \alpha_0\boldsymbol{A}_s\boldsymbol{h} = -\boldsymbol{C}\boldsymbol{e} - \boldsymbol{A}_s\sum_{i=1}^{N_p} g_i \tag{6}$$

$$\text{with} \quad \frac{\mathrm{d}g_i}{\mathrm{d}t} + \beta_i g_i = \alpha_i \boldsymbol{h}, \quad i = 1,\ldots,N_p, \tag{7}$$

where g_i are the auxilliary degrees of freedom in the SFVTD sense, i.e., corresponding to volume averages of the magnetic currents G_i on the face-staggered control volumes (cf. Fig. 1). The SIBC-operator, $\boldsymbol{A}_s$, in (6) turns out to be a diagonal matrix with diagonal entries given by the mesh face areas for faces lying on a SIBC surface and zero otherwise.

IV. Time Stepping

For the solution of (2),(6) and (7) in the time domain, the following time stepping strategy is applied. The discrete Ampere's and Faraday's equations, (2) and (6), respectively, are time-updated as usual using a leap-frog scheme. To preserve the stability bound of the original method, a semi-implicit approach is applied for the lossy SIBC term $\alpha_0\boldsymbol{A}_s\boldsymbol{h}$ appearing in (6) (cf. [5]). Finally, the set of ADEs (7) is solved with a second order accurate exponential time integrator [6]. The latter approach is known to provide optimal stability for stiff equations as is the case for the broadband SIBC-ADEs. Then, the overall time stepping scheme reads,

$$g_0^n = \alpha_0 \boldsymbol{h}^{n-1/2}, \tag{8}$$

$$g_i^n = g_i^{n-1}\boldsymbol{e}^{-\beta_i\Delta t} + \frac{\alpha_i}{\beta_i}\left(1 - \boldsymbol{e}^{-\beta_i\Delta t}\right)\boldsymbol{h}^{n-1/2}, \tag{9}$$

$$\boldsymbol{h}^{n+1/2} = \boldsymbol{h}^{n-1/2} - \Delta t\widetilde{\boldsymbol{M}}_\mu^{-1}\left(\boldsymbol{C}\boldsymbol{e} + \boldsymbol{A}_s\sum_{i=0}^{N_p} g_i^n\right), \tag{10}$$

$$\boldsymbol{e}^{n+1} = \boldsymbol{e}^n + \Delta t\boldsymbol{M}_\epsilon^{-1}\left(\boldsymbol{C}^T\boldsymbol{h}^{n+1/2} - \boldsymbol{j}^{n+1/2}\right), \tag{11}$$

where a modified magnetic mass matrix $\widetilde{\boldsymbol{M}}_\mu = \boldsymbol{M}_\mu + (L + \alpha_0\Delta t/2)\boldsymbol{A}_s$ is introduced. Note that $\widetilde{\boldsymbol{M}}_\mu$ is diagonal, so that its inverse can be readily computed.

V. Results

As a first validation test for the method, a cubical copper resonator with side length $l = 0.1$ mm is considered. A TM_{211}-mode ($\lambda = 81.65\mu$m, skin depth $\delta = 34$ nm) with $1J$ energy is initially excited in the resonator. These are typical values for the short range wakefields in electron accelerators.

Since a closed form analytical solution is not known, the semi-analytical result [7] for the damping factor of lossy cavity modes is used as a reference for investigating the accuracy of the SIBC formulation. Figure 2 (top) shows the total electromagnetic energy decaying in the cavity as a function of time normalized to oscillation period $T \approx 0.27ps$. The numerical accuracy and convergence rate of the cavity filling time with respect to the mesh resolution Δ is shown in Fig. 2 (bottom). Second order convergence rate is observed. In all simulations, the maximum possible time step matching exactly the 1D-CFL stability limit is used.

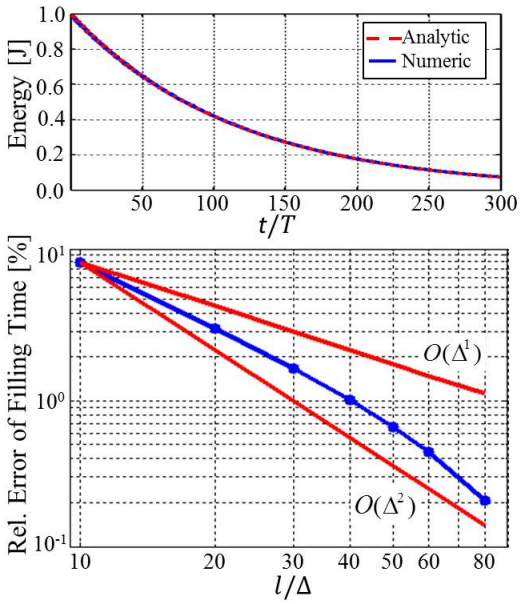

Figure 2: EM field energy vs. normalized time (top). Numerical convergence with respect to mesh resolution (bottom).

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